PREPARED FOR SUBMISSION TO JHEP

New coset matrix for $D = 6$ self-dual supergravity

Gérard Clément[a] **Dmitri V. Gal'tsov**[b]

[a] *Laboratoire de Physique Théorique LAPTH (CNRS), B.P.110, F-74941 Annecy-le-Vieux cedex, France*

[b] *Department of Theoretical Physics, Moscow State University, 119899, Moscow, Russia*

E-mail: gclement@lapp.in2p3.fr, galtsov@phys.msu.ru

ABSTRACT: Toroidal reduction of minimal six-dimensional supergravity, minimal five-dimensional supergravity and four-dimensional Einstein-Maxwell gravity to three dimensions gives rise to a sequence of cosets $O(4,3)/(O(4)\times O(3)) \supset G_{2(2)}/(SU(2)\times SU(2)) \supset SU(2,1)/S(U(2)\times U(1))$ which are invariant subspaces of each other. The known matrix representations of these cosets, however, are not suitable to realize these embeddings which could be useful for solution generation. We construct a new representation of the largest coset in terms of 7×7 real symmetric matrices and show how to select invariant subspaces corresponding to lower cosets by algebraic constraints. The new matrix representative may be also directly applied to minimal five-dimensional supergravity. Due to full $O(4,3)$ covariance it is simpler than the one derived by us previously for the coset $G_{2(2)}/(SU(2)\times SU(2))$.

KEYWORDS: gravity, supergravity, duality, symmetries

ARXIV EPRINT: 1301.5084

Contents

1 Introduction

The remarkable sequence of groups $O(7) \supset G_2 \supset SU(3)$ attracted attention in particle physics long ago. In a seminal paper Gunaydin and Gürsey [1] have given an extensive discussion of their properties, representations and applications to model building. Within the Lie algebra of $O(7)$, the subalgebras G_2 and $SU(3)$ form rather sophisticated closed structures which were explicitly given in [1] in terms of rotation generators of $O(7)$.

The maximally non-compact forms of the same groups $O(4,3) \supset G_{2(2)} \supset SU(2,1)$ play an important role in the gravity/supergravity context [2]. These group are hidden symmetries of six-dimensional minimal supergravity (MSG6) [3], five-dimensional minimal supergravity (MSG5) [4, 5] and four-dimensional Einstein-Maxwell (EM4) [6, 7] (super)gravity respectively, which are manifest as isometries of the target spaces of sigma models arising in their toroidal compactification to three dimensions [8–11]. More precisely, the compactified theories are gravity coupled scalar sigma models on the coset spaces $O(4,3)/(O(4) \times O(3))$, $G_{2(2)}/(SU(2) \times SU(2))$, $SU(2,1)/S(U(2) \times U(1))$ if the compactification tori are purely space-like, and $O(4,3)/(O(2,2) \times O(1,2))$, $G_{2(2)}/(SL(2,R) \times SL(2,R))$, $SU(2,1)/S(U(1,1) \times U(1))$ if one of the reduced dimensions is time. The last coset has been known for a long time as the manifold where the famous Ernst-Kinnersley-Mazur [6, 7, 12] symmetry operates. Its natural matrix representation is given in terms of 3×3 (pseudo)unitary matrices. The $G_{2(2)}$ coset was extensively explored recently as a tool for solution generation [5, 13–15] in MSG5. Fruitful for this purpose is the novel 7×7 matrix representation [5, 16] essentially related to the matrix representation of G_2 given by Gunaydin and Gürsey [1]. The coset $O(4,3)/(O(4) \times O(3))$ constitutes a particular case of the Hassan-Sen-Maharana-Schwarz (HSMS) cosets $O(n+p,n)/(O(n+p) \times O(n))$ arising in toroidal compactification of heterotic string effective theory, its matrix representation was given in [17–19]. In the case of $O(4,3)$ theory it is also realized in terms of 7×7 matrices. This representation, however, is rather complicated and not convenient to make contact with the sequence of subspaces $G_{2(2)}/(SU(2) \times SU(2))$ and $SU(2,1)/S(U(2) \times U(1))$ which can be useful in relating solutions of EM4, MSG5 and MSG6 theories between themselves.

The purpose of the present paper is to construct a new matrix representative of the coset $O(4,3)/(O(4) \times O(3))$ which allows for simple truncation to subspaces corresponding to MSG5 and EM4 theories. This is based again on the 7×7 representation, but with different parametrization of moduli. The new matrix is much simpler than the corresponding HSMS matrix and can be truncated to lower cosets by imposition of purely algebraic constraints. Our derivation is based on the direct toroidal reduction of MSG6 to three dimensions, explicit determination of target space isometry generators and subsequent exponentiation of the Borel subalgebra. We then extract the generators of the $G_{2(2)}$ and $SU(2,1)$ subgroups of $O(4,3)$ and derive algebraic constraints selecting the corresponding invariant subspaces of the coset $O(4,3)/(O(4) \times O(3))$ on which they act transitively.

2 $D = 6$ minimal supergravity

The bosonic action of six-dimensional minimal supergravity contains the metric and self-dual three-form field

$$S_{MSG6} = \int \left(\hat{R} - \frac{1}{12} \hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}} \hat{G}^{\hat{\mu}\hat{\nu}\hat{\lambda}} \right) \sqrt{-\hat{g}} d^6 x \,, \tag{2.1}$$

where $\hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}} \equiv 3\hat{C}_{[\hat{\mu}\hat{\nu},\hat{\lambda}]}$, with subsidiary condition

$$\hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}} = \frac{1}{6} \sqrt{-\hat{g}} \varepsilon_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}\hat{\sigma}\hat{\tau}} \hat{G}^{\hat{\rho}\hat{\sigma}\hat{\tau}} \,, \tag{2.2}$$

which has to be imposed after variation of the action.[1] The action (2.1) is a lowest-dimensional member of the even-dimensional sequence of actions containing self-dual form fields, the largest representative of which is the IIB ten-dimensional supergravity.

Somewhat unexpectedly, this action, being compactified on a circle, turns out to be non-locally dual to the truncated five-dimensional heterotic string effective action [17–19] which belongs to another sequence of the string actions. This can be hinted from the fact that the $D = 5$ heterotic string effective action truncated to the one-vector case gives rise to the $D = 3$ $O(4,3)/(O(4) \times O(3))$ coset theory (a particular case of the Sen's coset $O(d+1, d+1+p)/(O(d+1) \times O(d+1+p))$ where d is the number of compactified dimensions and p is the number of vector fields in the initial dimension [21]). Meanwhile the generic oxidation of the $O(4,3)/(O(4) \times O(3))$ coset has apart from the regular oxidation point $D = 5$ (which is the above heterotic effective action) also an anomalous *six-dimensional* oxidation point [3] which is just minimal $D = 6$ self-dual supergravity. This leads to a non-local duality between the two theories which can be made explicit as follows.

Denoting the coordinates $x^{\hat{\mu}} = (x^\mu, z)$ and assuming existence of the Killing vector ∂_z, we decompose the metric and the two-form potential as

$$ds_6^2 = e^{2\alpha\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{-6\alpha\phi} (dz + \mathscr{A}_\mu dx^\mu)^2 \,, \tag{2.3}$$

$$\hat{C} = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu + \frac{1}{\sqrt{2}} A_\nu dz \wedge dx^\nu \,, \tag{2.4}$$

where $\alpha^2 = 1/24$. The field equations are then equivalent to those derived from the five-dimensional action

$$S_5 = \int \left(R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{4} e^{4\alpha\phi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{12} e^{8\alpha\phi} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) \sqrt{-g_5} d^5 x \,, \tag{2.5}$$

with

$$\mathscr{F}_{\mu\nu} \equiv 2\mathscr{A}_{[\nu,\mu]} \,, \quad H^{\mu\nu\rho} \equiv -\frac{1}{2\sqrt{-g_5}} e^{-8\alpha\phi} \varepsilon^{\mu\nu\rho\sigma\tau} \mathscr{F}_{\sigma\tau} \,, \quad H_{\mu\nu\rho} = 3(B_{[\mu\nu,\rho]} + \frac{1}{2} F_{[\mu\nu} A_{\rho]}) \,. \tag{2.6}$$

[1] As in some other supergravity actions involving self-dual form fields, the quadratic action of the type (2.1) does not imply the self-duality condition (2.2), moreover it is zero, if self-duality is imposed in the action itself. One needs extra fields to construct a consistent action for chiral forms in a Lorentz-covariant way. We thank Dmitri Sorokin for drawing our attention to the references [20] where such an action for $D = 6$ minimal supergravity was presented. Here we deal with classical equations of motion, so it will be sufficient to impose the condition (2.2) by hand after variation is performed. The dimensional reduction of the full action [20] is more involved, but this does not change the results on the classical level.

This is a heterotic string type effective action [19, 21] with one vector and one antisymmetric second rank tensor fields. Note that the Maxwell field $F_{\mu\nu}$ in this action originates from the six-dimensional three-form, while the five-dimensional three-form $H_{\mu\nu\rho}$ is obtained by dualisation of the Kaluza-Klein two-form. Therefore the relation between the six and five-dimensional metrics and matter fields is non-local.

Due to this duality one can reduce the six-dimensional action (2.1) (which is the subject of the present paper) to three dimensions along two different compactification schemes. The first consists in using the well-studied compactification of the corresponding five-dimensional heterotic string action (2.5) along the lines of [19, 21]. The second, suggested in the present paper, consists in direct compactification of the initial six-dimensional action (2.1) on a three-torus.

The first way which we briefly sketch here gives Sen's type representation for the coset matrix [21]. Splitting the coordinates as z^a, $x^\mu = x^i$, $a = 1,2$, $i = 1,2,3$, with z^a along the compactified dimensions, we parameterize the metric and the matter fields as

$$\begin{aligned}
&ds^2 = \lambda_{ab}(dz^a + A^a_i dx^i)(dz^b + A^b_j dx^j) + \tau^{-1} h_{ij} dx^i dx^j\,, \qquad \tau = -\det\lambda\,,\\
&A_\mu dx^\mu = \psi_a(dz^a + A^a_i dx^i) - A^5_i dx^i\,, \qquad\qquad (2.7)\\
&B_{\mu\nu} dx^\mu \wedge dx^\nu = B_{ab}(dz^a + A^a_i dx^i)\wedge(dz^b + A^b_j dx^j) + (A_{i(a+2)} - \frac{1}{2}\psi_a A^5_i)(dz^a\wedge dx^i - dx^i \wedge dz^a)\\
&+(B_{ij} + A^a_{[i}A_{j](a+2)})dx^i\wedge dx^j\,.
\end{aligned}$$

The three-dimensional reduced action can be presented in terms of the matrix sigma model

$$S_3 = \int \left\{ R_3(h) - \frac{1}{8}\mathrm{Tr}\left[(\partial_i \mathscr{M})\mathscr{M}^{-1}(\partial_j \mathscr{M})\mathscr{M}^{-1}\right] h^{ij}\right\}\sqrt{h}d^3x. \qquad (2.8)$$

According to [21], the coset matrix $\mathscr{M}$ is constructed in three steps: first one defines of the matrix M of non-dualized moduli, then dualisation of three-dimensional vectors to scalar potentials is performed, and finally the matrix $\mathscr{M}$ is constructed in terms of M and the new scalars. To built the moduli matrix M one arranges the five vector fields in a column matrix A^A_i ($A = 1,\ldots,5$),

$$A^A_i = (A^a_i, A_{i(a+2)}, A^5_i)\,, \qquad (2.9)$$

with the field strengths

$$F^A_{ij} \equiv 2\partial_{[i}A^A_{j]}\,, \quad H_{ijk} = 3(\partial_{[i}B_{jk]} + \frac{1}{2}A^A_{[i}L_{AB}F^B_{jk]})\,, \qquad (2.10)$$

where L is the 5×5 matrix written in block form

$$L = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1 \end{pmatrix}. \qquad (2.11)$$

The 2-form B_{ij} is actually fixed by the gauge condition

$$H_{ijk} = 0\,. \qquad (2.12)$$

The 5×5 moduli matrix M_{AB} then reads, in block form,

$$M = \begin{pmatrix} \gamma^{-1} & \gamma^{-1}C & \gamma^{-1}\psi \\ C^T\gamma^{-1} & (\gamma+C^T)\gamma^{-1}(\gamma+C) & (\gamma+C^T)\gamma^{-1}\psi \\ \psi^T\gamma^{-1} & \psi^T\gamma^{-1}(\gamma+C) & 1+\psi^T\gamma^{-1}\psi \end{pmatrix}, \tag{2.13}$$

where $\gamma_{ab} \equiv e^{-\nu\phi}\lambda_{ab}$ ($\nu = \sqrt{2/3}$), and C is the 2×2 matrix $C = B + \frac{1}{2}\psi\psi^T$. The matrix M is symmetric, and satisfies

$$MLM^T = L\,. \tag{2.14}$$

The next step involves dualisation of the three–dimensional vector fields according to

$$\tau\sqrt{h}e^{\nu\phi}h^{ii'}h^{jj'}(ML)_{AB}F^B_{i'j'} = \varepsilon^{ijk}\partial_k\omega_A\,, \tag{2.15}$$

defines the row matrix

$$\omega \equiv (\overline{\omega}^a, \omega_a, \omega_5)\,. \tag{2.16}$$

Now, using the result of [21] it is straightforward to write down the 7×7 matrix $\mathscr{M}$ in a block form:

$$\mathscr{M} = \begin{pmatrix} M + e^{-\nu\phi}\omega\omega^T & -e^{-\nu\phi}\omega^T & ML\omega^T + \frac{1}{2}e^{-\nu\phi}\omega^T(\omega L\omega^T) \\ -e^{-\nu\phi}\omega & e^{-\nu\phi} & -\frac{1}{2}e^{-\nu\phi}(\omega L\omega^T) \\ \omega LM + \frac{1}{2}e^{-\nu\phi}\omega(\omega L\omega^T) & -\frac{1}{2}e^{-\nu\phi}(\omega L\omega^T) & e^{\nu\phi} + \omega LML\omega^T + \frac{1}{4}e^{-\nu\phi}(\omega L\omega^T)^2 \end{pmatrix}. \tag{2.17}$$

Thus, in principle, the Sen's matrix can be also used in the case of $D=6$ minimal supergravity not belonging to the sequence of the heterotic string effective actions. But disadvantage of such an approach, apart from relative complexity of the matrix (2.17), lies in the fact that the variables of the five-dimensional heterotic action in terms of which this representation is written, are still non-trivially related to the initial six-dimensional variables. Another desired feature which can be demanded from the coset representation of the $D=6$ theory is the possibility of its simple truncation to five-dimensional minimal supergravity whose $D=3$ coset $G_{2(2)}/(SU(2)\times SU(2))$ is an invariant subspace of the coset $O(4,3)/(O(4)\times O(3))$. This can be achieved using the direct toroidal compactification of $D=6$ minimal supergravity to three dimensions. Before doing this, we briefly review the coset structure of five-dimensional minimal supergravity. In both cases we will use the technique applied in [5] which consists in i) obtaining an explicit form form of the target space metric, ii) identifying its isometry algebra, iii) exponentiating the Borel subalgebra to get suitable matrix representation. Though technically different, it is conceptually the same construction as used by Maharana-Schwarz and Sen [18, 21].

3 $D=5$ minimal supergravity

The action of MSG5 reads

$$S_{MSG5} = \int \left(\left[R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right]\sqrt{g_5} - \frac{1}{12\sqrt{3}}\varepsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda\right)d^5x\,, \tag{3.1}$$

with $F = dA$. We compactify on a two-torus using

$$ds_5^2 = \lambda_{ab}(dz^a + a_i^a dx^i)(dz^b + a_j^b dx^j) + \tau^{-1} h_{ij} dx^i dx^j \,, \tag{3.2}$$

$$A_{(5)\mu} dx^\mu = \sqrt{3}(\psi_a dz^a + A_i dx^i)\,, \tag{3.3}$$

where $a, b = 0, 1$ and $\tau \equiv |\det\lambda|$. The $\nu = i$ components of the Maxwell-Chern-Simons equations allow to dualize the vector magnetic potential A_i to a scalar magnetic potential μ defined by

$$F^{ij} = a^{aj}\partial^i\psi_a - a^{ai}\partial^j\psi_a + \frac{1}{\tau\sqrt{h}}\varepsilon^{ijk}\eta_k \,, \qquad \eta_k = \partial_k\mu + \varepsilon^{ab}\psi_a\partial_k\psi_b \,. \tag{3.4}$$

Similarly, the $\mu = i$, $\nu = a$ components of the Einstein equations are integrated by

$$\lambda_{ab} G^{bij} = \frac{1}{\tau\sqrt{h}}\varepsilon^{ijk} V_{ak} \,, \qquad V_{ak} = \partial_k\omega_a - \psi_a\left(3\partial_k\mu + \varepsilon^{bc}\psi_b\partial_k\psi_c\right) \,, \tag{3.5}$$

where $G^b = da^b$, and ω_a is the 'twist' or gravimagnetic two-potential. The $D = 3$ sigma model

$$S_3 = \int \left(R_3(h) - \frac{1}{2} G_{AB}\partial_i\Phi^A\partial_j\Phi^B h^{ij}\sqrt{h}\right) d^3x \,, \tag{3.6}$$

is then obtained with eight target space coordinates $\Phi^A = \{\lambda_{ab}, \omega_a, \psi_a, \mu\}$ and metric

$$dl^2 = G_{AB} d\Phi^A d\Phi^B = \frac{1}{2}\mathrm{Tr}(\lambda^{-1} d\lambda \lambda^{-1} d\lambda) + \frac{1}{2}\tau^{-2} d\tau^2 - \tau^{-1} V^T \lambda^{-1} V + 3\left(d\psi^T \lambda^{-1} d\psi - \tau^{-1}\eta^2\right) \,. \tag{3.7}$$

This space has 14 Killing vectors which were determined in terms of these variables in [5, 16]. Nine manifest infinitesimal symmetries (or generalised gauge transformations), grouped according to their transformations under $GL(2R)$ (the group of linear transformations in the (z^1, z^2) plane) into the quadruplet

$$M_a{}^b = 2\lambda_{ac}\frac{\partial}{\partial\lambda_{cb}} + \omega_a\frac{\partial}{\partial\omega_b} + \delta_a^b\omega_c\frac{\partial}{\partial\omega_c} + \psi_a\frac{\partial}{\partial\psi_b} + \delta_a^b\mu\frac{\partial}{\partial\mu} \tag{3.8}$$

(the generators of the $gl(2, R)$ subalgebra), the doublet and the singlet associated with the the three cyclic 'magnetic' coordinates:

$$N^a = \frac{\partial}{\partial\omega_a} \,, \quad Q = \frac{\partial}{\partial\mu} \,, \tag{3.9}$$

and the doublet generating infinitesimal gauge transformations of the ψ_a

$$R^a = \frac{\partial}{\partial\psi_a} + 3\mu\frac{\partial}{\partial\omega_a} - \varepsilon^{ab}\psi_b\left(\frac{\partial}{\partial\mu} + \psi_c\frac{\partial}{\partial\omega_c}\right) \,. \tag{3.10}$$

The five remaining, non trivial infinitesimal isometries L_a, P_a and T closing the Lie algebra $g_{2(2)}$ are more complicated, their full expression is given in [16]. The L_a, $M_a{}^b$ and N^a generate the vacuum subalgebra $sl(3, R)$. Assuming a spacelike two-torus, the target space (3.7) is identified as the coset space $G_{2(2)}/(SU(2) \times SU(2))$.

The 7×7 symmetric matrix representative of the coset obtained by exponentiation of the Borel subalgebra [5, 16] exhibits a highly nonlinear dependence on the moduli. Its structure is quite different from that of the Sen matrix (2.17) for the coset $O(4, 3)/(O(4) \times O(3))$, so it is practically impossible to relate them.

4 New representative for $D = 6$ minimal supergravity

A simpler representation of the coset $O(4,3)/(O(4)\times O(3)$ may be achieved by performing direct compactification of the six-dimensional theory on T^3. We start with the Lagrangian (2.1), and assume 3 Killing vectors ∂_a ($a = 1,2,3$). The six-dimensional metric and 3-form may be parameterized by

$$\begin{aligned} ds_6^2 &= \lambda_{ab}(dz^a + a_i^a dx^i)(dz^b + a_j^b dx^j) + \tau^{-1} h_{ij} dx^i dx^j \,, \\ \hat{G}_{abc} &= 0 \,, \quad \hat{G}_{abi} = \hat{B}_{ab,i} \,, \end{aligned} \tag{4.1}$$

($\tau \equiv -\det\lambda$, $i,j = 4,5,6$) and the 10 remaining components of $\hat{G}$ related to these by self-duality. Put

$$\hat{B}_{ab} \equiv \varepsilon_{abc}\chi^c \,. \tag{4.2}$$

Then,

$$\hat{G}_{abi} = \varepsilon_{abc}\chi^c_{,i} \,, \quad \hat{G}^{aij} = -\frac{\tau}{\sqrt{h}}\varepsilon^{ijk}\chi^a_{,k} \,. \tag{4.3}$$

The mixed Einstein equations

$$\begin{aligned} \hat{R}_a^i &\equiv \frac{\tau}{2\sqrt{h}}\partial_j[\tau\sqrt{h}\lambda_{ab}\mathscr{F}^{bij}] \\ &= \frac{1}{2}\hat{G}^{ibj}\hat{G}_{abj} = \frac{\tau}{2\sqrt{h}}\partial_j[\varepsilon^{ijk}\varepsilon_{abc}\chi^b_{,k}\chi^c] \end{aligned} \tag{4.4}$$

($\mathscr{F}^b \equiv da^b$) are solved by

$$\lambda_{ab}\mathscr{F}^{bij} = \frac{1}{\tau\sqrt{h}}\varepsilon^{ijk}V_{ak} \,, \quad V_{ak} \equiv \partial_k\omega_a + \varepsilon_{abc}\chi^b_{,k}\chi^c \,. \tag{4.5}$$

The remaining Einstein equations then lead to the gravitating sigma model with target space metric

$$dl^2 = \frac{1}{2}\mathrm{Tr}(\lambda^{-1}d\lambda\lambda^{-1}d\lambda) + \frac{1}{2}\tau^{-2}d\tau^2 - \tau^{-1}V^T\lambda^{-1}V - 2\tau^{-1}d\chi^T\lambda d\chi \,, \tag{4.6}$$

where

$$V \equiv d\omega - \chi\wedge d\chi \,. \tag{4.7}$$

The dimension of this target space is twelve: six components of the symmetric matrix λ_{ab} and two triplets ω_a, χ^a. In Appendix A we check that it admits 21 Killing vectors generating the Lie algebra $o(4,3)$. These include nine Killing vectors ${M_a}^b$ generating the algebra $gl(3,R)$ of linear transformations in the three-Killing vector space, six vectors N^a and L_a which together with the ${M_a}^b$ generate the isometry algebra $sl(4,R)$ for the target subspace corresponding to the six-dimensional vacuum sector, and six more vectors R_a and P^a which complete the algebra $o(4,3)$. The fifteen Killing vectors ${M_a}^b$, N^a and R_a generate generalized gauge transformations, with the N^a generating translations of the twists ω_a and the R_a generating gauge transformations of the χ^a.

In Appendix B we construct real matrix representatives of $o(4,3)$, beginning with the subalgebra $o(3,3)\sim sl(4,R)$. Rather than using the Maison parametrization [22] of $sl(4,R)$ in terms of 4×4 matrices (which presumably would lead to a representation of $o(4,3)$ in terms of 8×8 matrices), we use the representation of $o(3,3)$ in terms of 6×6 matrices. These are then promoted

to 7×7 matrices by the addition of a row and a column, and completed by six 7×7 matrices R_a and P^a closing the algebra $o(4,3)$. The 7×7 coset matrix representative is then constructed in a standard fashion as

$$\mathcal{M} = \mathcal{N}^T \eta \mathcal{N} \,, \tag{4.8}$$

where $\mathcal{N}$ is obtained by exponentiating a suitable Borel subalgebra of $0(4,3)$, and η is a suitably chosen constant matrix. The resulting coset representative is, in block form,

$$\mathcal{M} = \begin{pmatrix} \mu & \sqrt{2}\mu\chi & \mu\gamma \\ \sqrt{2}\chi^T\mu & -1+2\chi^T\mu\chi & \sqrt{2}(\chi^T\mu\gamma+\tilde{\chi}) \\ \gamma^T\mu & \sqrt{2}(\gamma^T\mu\chi+\tilde{\chi}^T) & \gamma^T\mu\gamma-2\tilde{\chi}^T\tilde{\chi}+\tilde{\mu}^{-1} \end{pmatrix} \tag{4.9}$$

where $\tilde{}$ denotes the anti-transposition, i.e. transposition relative to the anti- (or minor) diagonal, and

$$\mu = \tau^{-1}\lambda = \tau^{-1} \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} ,$$

$$\chi = \begin{pmatrix} \chi^1 \\ \chi^2 \\ \chi^3 \end{pmatrix} , \quad \tilde{\chi} = (\chi_3, \chi_2, \chi_1) , \quad \gamma = \hat{\omega} - \chi\tilde{\chi} , \quad \hat{\omega} = \begin{pmatrix} -\omega_2 & \omega_3 & 0 \\ \omega_1 & 0 & -\omega_3 \\ 0 & -\omega_1 & \omega_2 \end{pmatrix} . \tag{4.10}$$

One can check that the target space metric (4.6) can be expressed as

$$dl^2 = \frac{1}{4}\mathrm{Tr}(\mathcal{M}^{-1}d\mathcal{M}\mathcal{M}^{-1}d\mathcal{M}) . \tag{4.11}$$

In the case of a Lorentzian six-dimensional space E_6 with signature $(-+++++)$ and an Euclidean reduced three-space (so that one of the Killing vectors of E_6 is timelike), the symmetric target space $\mathcal{T}$ of metric (4.6) is the coset $G/H = O(4,3)/O(2,2)\times O(2,1)$. H is the isotropy group leaving invariant any given point of the target space, which may be chosen to be the point at infinity of $\mathcal{T}$. Thus it is relevant to examine the various possible asymptotic behaviors for asymptotically flat six-dimensional configurations.

Minkowski asymptotics. For an asymptotically Minkowskian metric, or for a metric which is asymptotically the product of a four-dimensional black hole by a 2-torus, with x^1 the time coordinate, the asymptotic coset representative is

$$\mathcal{M}_\infty = \eta_M = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} . \tag{4.12}$$

This asymptotic behavior is preserved by the nine Killing vectors

$$\begin{aligned} \bar{X}_1 &= -{M_2}^3 + {M_3}^2 \,, & \bar{X}_2 &= {M_3}^1 + {M_1}^3 \,, & \bar{X}_3 &= -{M_1}^2 - {M_2}^1 \,, \\ \bar{Y}_1 &= N^1 + L_1 \,, & \bar{Y}_2 &= N^2 - L_2 \,, & \bar{Y}_3 &= N^3 - L_3 \,, \\ \bar{Z}_1 &= P^1 - R_1 \,, & \bar{Z}_2 &= P^2 + R_2 \,, & \bar{Z}_3 &= P^3 + R_3 \end{aligned} \tag{4.13}$$

(with the first three pure gauge), satisfying the commutation relations

$$[\overline{X}_a,\overline{X}_b] = [\overline{Y}_a,\overline{Y}_b] = \varepsilon_{abc}\eta_c\overline{X}_c\,, \quad [\overline{Z}_a,\overline{Z}_b] = 2\varepsilon_{abc}\eta_c(\overline{X}_c+\overline{Y}_c)\,,$$
$$[\overline{X}_a,\overline{Y}_b] = \varepsilon_{abc}\eta_c\overline{Y}_c\,, \quad [\overline{Y}_a,\overline{Z}_b] = [\overline{Z}_a,\overline{X}_b] = \varepsilon_{abc}\eta_c\overline{Z}_c\,, \tag{4.14}$$

with $\eta_1 = -1, \eta_2 = \eta_3 = +1$. The combinations

$$K^0_a = \frac{1}{2}(\overline{X}_a - \overline{Y}_a)\,, \quad K^\pm_a = \frac{1}{4}(\overline{X}_a + \overline{Y}_a \pm \overline{Z}_a) \tag{4.15}$$

generate three commuting copies of the Lie algebra of $O(2,1)$,

$$[K_a,K_b] = \varepsilon_{abc}\eta_c K_c\,. \tag{4.16}$$

We thus recover the isotropy subgroup $H = O(2,2)\times O(2,1) = O(2,1)^3$.

Black string asymptotics. The static Myers-Perry (Tangherlini) six-dimensional black string (the product of a five-dimensional black hole by a circle) is

$$ds_6^2 = -(1-m/r^2)dt^2 + \frac{dr^2}{1-m/r^2} + \frac{r^2}{4}\left[(d\eta - \cos\theta d\varphi)^2 + d\theta^2 + \sin^2\theta d\varphi^2\right] + (d\zeta)^2\,. \tag{4.17}$$

This has four commuting Killing vectors. Reduction relative e.g. to $\partial_1 = \partial_t$, $\partial_2 = \partial_\eta$ and $\partial_3 = \partial_\zeta$ leads to

$$\lambda \;=\; \mathrm{diag}[-(r^2-m)/r^2,\; r^2/4,\; 1]\,, \quad \tau = \frac{r^2-m}{4}\,, \quad a^2_\varphi = -\cos\theta\,,$$
$$d\sigma^2 \;\equiv\; h_{ij}dx^i dx^j = \frac{r^2}{4}\left[dr^2 + \frac{r^2-m}{4}(d\theta^2 + \sin^2\theta d\varphi^2)\right]\,, \tag{4.18}$$

leading to

$$\omega_2 = \frac{r^2+c}{4}\,, \tag{4.19}$$

with c a constant of integration. Computation of the asymptotic behavior of the lower right-hand side 3×3 block in (4.9) gives

$$\tau^{-1}\gamma^T\lambda\gamma + \tau\tilde{\lambda}^{-1} \underset{r\to\infty}{\simeq} \mathrm{diag}[-(m+2c)/4,\; 1,\; (m+2c)/4]\,, \tag{4.20}$$

which is equal to the asymptotic behavior of the upper left-hand side block for the value $c = -m/2$. In this case,

$$\mathscr{M}_\infty = \begin{pmatrix} 0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1 \\ 0&0&0&-1&0&0&0 \\ 1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0 \end{pmatrix}\,. \tag{4.21}$$

This asymptotic behavior is preserved by the nine Killing vectors

$$X_+ = -M_2{}^1 - L_1\,, \qquad X_0 = M_2{}^2\,, \qquad X_- = -M_1{}^2 + N^1\,,$$
$$Y_+ = M_2{}^3 - L_3\,, \quad Y_0 = M_1{}^3 + M_3{}^1\,, \quad Y_- = M_3{}^2 + N^3\,,$$
$$Z_+ = -P^1 + P^3\,, \qquad Z_0 = P^2 + R_2\,, \qquad Z_- = -R_1 + R_3\,. \tag{4.22}$$

X_0, X_-, Y_0, Y_- and Z_- are pure gauge. The first three generate an $SL(2,R)$,

$$[X_0, X_\pm] = \pm X_\pm\,, \quad [X_+, X_-] = X_0\,, \tag{4.23}$$

or symbolically $[X,X] = X$. The full algebra

$$\begin{aligned}
&[X,X] = [Y,Y] = X\,, \quad [Z,Z] = 2(X+Y)\,,\\
&[X,Y] = [Y,X] = Y\,, \quad [X,Z] = [Z,X] = [Y,Z] = [Z,Y] = Z\,,
\end{aligned} \tag{4.24}$$

(with commutators such as $[X_0,Y_0]$ and $[X_\pm,Y_\pm]$ vanishing) can be split, as in the case of Minkowski asymptotics, into three commuting $sl(2,R) = O(2,1)$ generated by the combinations

$$J^0 = \frac{1}{2}(X-Y)\,, \quad J^\pm = \frac{1}{4}(X+Y\pm Z)\,, \tag{4.25}$$

so that the isotropy subgroup $H = O(2,2)\times O(2,1)$ is again recovered.

Black hole asymptotics. The static six-dimensional black hole

$$ds_6^2 = -(1-m/r^3)dt^2 + \frac{dr^2}{1-m/r^3} + r^2\left[d\theta^2 + \cos^2\theta d\zeta^2 + \sin^2\theta(d\eta^2 + \sin^2\eta d\varphi^2)\right] \tag{4.26}$$

has only three commuting Killing vectors ∂_t, ∂_ζ and ∂_φ. Reduction relative to these vectors leads to

$$\begin{aligned}
\mu &= \mathrm{diag}[-4/r^4\sin^2 2\theta\sin^2\eta,\, r/(r^3-m)\sin^2\theta\sin^2\eta,\, r/(r^3-m)\cos^2\theta]\,.\\
\chi &= \gamma = 0\,.
\end{aligned} \tag{4.27}$$

The resulting matrix $\mathcal{M}$ has no regular limit at spatial infinity. As in the well-known case of four-dimensional Einstein-Maxwell theory reduced relative to the azimuthal Killing vector [23], new solutions can be generated from this by $O(4,3)$ transformations, but these will be always non-asymptotically flat.

5 Invariant subspaces

The new matrix (4.9) parameterizes the twelve-dimensional coset space of MSG6 theory. It may be also applied as a representative of the embedded eight-dimensional coset $G_{2(2)}/(SU(2)\times SU(2)$ corresponding to MSG5 and four-dimensional coset $SU(2,1)/S(U(2)\times U(1))$ corresponding to EM4. These may be selected by purely algebraic constraints on the potentials. To find these constraints one has to consider dimensional reductions and consistent truncations which relate these theories to MSG6.

5.1 $D = 5$ minimal supergravity

The coset $G_{2(2)}/(SU(2)\times SU(2)$ is a totally geodesic subspace of the coset $O(4,3)/(O(4)\times O(3))$, so MSG6 compactified on a circle can be consistently truncated to MSG5. Indeed, it can be checked [24] that the equations of motion following from (2.5) are consistent with the constraints

$$\phi = 0\,, \qquad F_{\mu\nu} = \mp\frac{\sqrt{-g_5}}{3\sqrt{2}}\varepsilon_{\mu\nu\rho\sigma\tau}H^{\rho\sigma\tau}\,, \tag{5.1}$$

in which case they reduce to those of MDG5, provided the two-form field $F_{\mu\nu}$ is rescaled by

$$F_{\mu\nu} \to \pm\sqrt{\frac{2}{3}}F_{\mu\nu}\,. \tag{5.2}$$

Note that, in view of the second relation in (2.6), the second contraint (5.1) is equivalent to identification of the Maxwell two-form and the Kaluza-Klein two-form of the reduced theory (2.5), in which case the dilaton can be consistently set to zero. We must now identify the corresponding constraints in terms of the target space variables. Inspecting the definitions of the three-dimensional target space variables (4.1)-(4.5) one finds

$$\lambda_{33} = 1\,, \quad \lambda_{a3} = \mp\varepsilon_{ab}\chi^b\,, \quad \omega_3 = \mp\chi^3\,. \tag{5.3}$$

Since the G_2 sector arises as a consistent truncation of the five-dimensional reduction of the original six-dimensional model, it is not surprising that in the reduction to three dimensions the three-covariance is broken down to two-covariance. Actually, knowing this (and making some educated guesses) is enough to find the two g_2 subalgebras generated by the Killing vectors preserving the constraints (5.3). In the two-covariant notation of Sect. 4 of [5], their generators are related to those of $o(4,3)$ by

$$\begin{array}{rcl}
\underline{g_2} & & \underline{o(4,3)} \\
{M_a}^b & = & {M_a}^b \\
N^a & = & N^a \\
L_a & = & L_a \\
Q & = & -N^3 \pm R_3 \\
T & = & -L_3 \mp P^3\,, \\
R^a & = & -{M_3}^a \pm \varepsilon^{ab}R_b \\
P_a & = & -{M_a}^3 \pm \varepsilon_{ab}P^b
\end{array} \tag{5.4}$$

with $a,b = 1,2$. It is easy to check that these combinations satisfy the commutation relations (92)-(97) of [5].

Conversely, comparing the covariant three-dimensional reductions of MSG5 and MSG6, we see that any solution $(ds^2_{(5)}, A_{(5)})$ of MSG5 with two commuting Killing vectors can be oxidized to a solution of MSG6 with three commuting Killing vectors given by

$$\begin{aligned}
ds_{(6)}{}^2 &= ds^2_{(5)} + (-\psi_a dx^a + dz - A_i dx^i)^2\,, \\
\chi &= \pm(-\psi_2,\, \psi_1,\, \mu)\,.
\end{aligned} \tag{5.5}$$

It follows that, given a solution of MSG5 with two commuting Killing vectors, one can generate from this a new solution by going through the following steps: 1) oxidize the seed solution to a solution of MSG6 by (5.5); 2) construct its coset representative (4.9); 3) transform this,

$$\mathscr{M}' = P^T \mathscr{M} P\,, \tag{5.6}$$

by the action of an $O(4,3)$ transformation P generated by the generators of the second column of (5.4); 4) extract from $\mathscr{M}'$ the new solution of MSG6; 5) reduce this to five dimensions by (5.5). In view of the simplicity of the matrix representation (4.9) compared to that previously

known for MSG5, this procedure might be easier to implement than direct generation by $G_{2(2)}$ transformations.

The generators preserving both five-dimensional Myers-Perry (or black string) asymptotics *and* G_2 truncation are

$$J^{\mp} = \frac{1}{4}(X+Y\mp Z)\,, \quad J^0+J^{\pm} = \frac{1}{4}(3X-Y\pm Z)\,, \tag{5.7}$$

generating two commuting copies of $sl(2,R) = o(2,1)$. The non-trivial generators are

$$\begin{aligned} \pm G_{0(\pm)} &= Z_0 \mp Y_0 = P^2 + R_2 \mp (M_1{}^3 + M_3{}^1) \qquad (\text{electric charge})\,, \\ \pm G_{+(\pm)} &= Z_+ \mp Y_+ = -P^1 + P^3 \mp (M_2{}^3 - L_3) \qquad (\text{two dipole charges})\,, \\ F_+ &= X_+ = -M_2{}^1 - L_1 \qquad (\text{angular momentum})\,. \end{aligned} \tag{5.8}$$

Clearly, all the generators (4.22) preserving black string asymptotics are linear combinations of the four non-trivial generators F_+, $G_{+(+)}$, $G_{+(-)}$, and one of the $G_{0(\pm)}$, together with gauge transformations. These four generators applied to a black string will do the same job as the corresponding G_2 generators (and in particular, preserve the black string condition $\lambda_{33} = 1$), but in a simpler fashion. The non-vanishing commutators between these four generators are

$$\begin{aligned} \left[G_{0(\pm)}, F_+\right] &= G_{+(\pm)}\,, \\ \left[G_{0(\pm)}, G_{+(\pm)}\right] &= 3F_+ - 2G_{+(\pm)}\,, \\ \left[G_{0(\pm)}, G_{+(\mp)}\right] &= F_+ - G_{+(+)} - G_{+(-)}\,. \end{aligned} \tag{5.9}$$

5.2 $D = 4$ Einstein-Maxwell

To identify the constraints selecting the $SU(2,1)/S(U(2)\times U(1))$ subspace of the G_2 coset, one must first compactify MSG5 on a circle[5], since this subcoset corresponds to four-dimensional Einstein-Maxwell theory. Assuming the existence of a space-like Killing vector ∂_z, we parametrize the five-dimensional metric and Maxwell field by

$$ds_5^2 = \mathrm{e}^{-2\phi}(dz + C_\mu dx^\mu)^2 + \mathrm{e}^{\phi} ds_4^2, \tag{5.10}$$

$$A_5 = A_\mu dx^\mu + \sqrt{3}\kappa dz, \tag{5.11}$$

($\mu = 1\ldots4$). The corresponding four-dimensional action

$$S_4 = \int d^4x\sqrt{-g}\left[R - \frac{3}{2}(\partial\phi)^2 - \frac{3}{2}\mathrm{e}^{2\phi}(\partial\kappa)^2 - \frac{1}{4}\mathrm{e}^{-3\phi}G^2 - \frac{1}{4}\mathrm{e}^{-\phi}\tilde{F}^2 - \frac{1}{2}\kappa F F^*\right], \tag{5.12}$$

where

$$G = dC, \quad F = dA, \quad \tilde{F} = F + \sqrt{3}C\wedge d\kappa, \tag{5.13}$$

and F^* is the four-dimensional Hodge dual of F, describes an Einstein theory with two coupled abelian gauge fields F and G, a dilaton ϕ and an axion κ. The field equations in terms of the four-dimensional variables read

$$\nabla^2\phi - \mathrm{e}^{2\phi}(\partial\kappa)^2 + \frac{1}{4}\mathrm{e}^{-3\phi}G^2 + \frac{1}{12}\mathrm{e}^{-\phi}\tilde{F}^2 = 0, \tag{5.14}$$

$$\nabla_\mu\left(\mathrm{e}^{2\phi}\nabla^\mu\kappa\right) - \frac{1}{3}\left[\sqrt{3}\nabla_\mu(\mathrm{e}^{-\phi}\tilde{F}^{\mu\nu}C_\nu) + \frac{1}{2}F_{\mu\nu}F^{*\mu\nu}\right] = 0, \tag{5.15}$$

$$\nabla_\mu\left(\mathrm{e}^{-\phi}\tilde{F}^{\mu\nu} + 2\kappa F^{*\mu\nu}\right) = 0, \tag{5.16}$$

$$\nabla_\mu\left(\mathrm{e}^{-3\phi}G^{\mu\nu}\right) + \sqrt{3}\mathrm{e}^{-\phi}\tilde{F}^{\mu\nu}\partial_\mu\kappa = 0. \tag{5.17}$$

Truncation to the Einstein-Maxwell system is achieved by imposing

$$\phi = 0, \quad \kappa = 0, \quad G_{\mu\nu} = \frac{1}{2\sqrt{3}}\sqrt{g_4}\varepsilon_{\mu\nu\rho\sigma}F_{(4)}^{\rho\sigma}. \tag{5.18}$$

After reduction to three dimensions, this leads to the constraints

$$\lambda_{22} = 1, \quad \psi_2 = 0, \quad \lambda_{12} = \mu, \quad \omega_2 = -\psi_1. \tag{5.19}$$

We find that these constraints are preserved by the eight infinitesimal transformations

$$\begin{gathered} K_1 = {M_1}^1, \\ K_2 = {M_2}^1 + Q, \; K_3 = {M_1}^2 - T, \\ K_4 = N^1, \qquad K_5 = L_1, \\ K_6 = N^2 - R^1, \; K_7 = L_2 + P_1, \\ K_8 = P_2 - R^2. \end{gathered} \tag{5.20}$$

From the commutation relations given in [16], we find that the K_M ($M = 1, \ldots, 8$) generate the Lie algebra of $SU(2,1)$, which may be put in the Cartan-Weyl form [25], with

$$\begin{gathered} H_1 = \tfrac{1}{2\sqrt{3}}K_1, \quad H_2 = \tfrac{i}{6}K_8, \\ \begin{aligned} &E_1 = \tfrac{1}{\sqrt{6}}K_5, && E_- = -\tfrac{1}{\sqrt{6}}K_4, && \alpha_1 = \tfrac{1}{\sqrt{3}}(1,0), \\ &E_2 = \tfrac{1}{4\sqrt{3}}(-K_6 + iK_2), && E_{-2} = \tfrac{1}{4\sqrt{3}}(K_7 - iK_3), && \alpha_2 = \tfrac{1}{\sqrt{3}}\left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right), \\ &E_3 = \tfrac{1}{4\sqrt{3}}(K_3 - iK_7), && E_{-3} = \tfrac{1}{4\sqrt{3}}(K_2 - iK_6), && \alpha_3 = \tfrac{1}{\sqrt{3}}\left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right). \end{aligned} \end{gathered} \tag{5.21}$$

6 Outlook

The main result of this paper is the new representative (4.9) of the coset $O(4,3)/(O(4)\times O(3))$ which is a symmetric 7×7 matrix, given in block form. This matrix is substantially simpler than the matrix (2.17) constructed by Sen's method. Moreover, it also looks simpler than the G_2 matrix for the coset $G_{2(2)}/(SU(2)\times SU(2))$ constructed by us previously [5] and used for solution generation in [5, 13]. The reason is that constraining to the subspace $G_{2(2)}/(SU(2)\times SU(2))$ one loses the $O(4,3)$ covariance which simplifies the underlying matrix structure. After truncation to vacuum five-dimensional gravity ($\chi^a = 0$), the new matrix leads to a 7×7 matrix representative of the coset $SL(3,R)/O(2,1)$ which is different from that resulting from the truncation to vacuum gravity ($\psi_a = \mu = 0$) of our previous G_2 matrix [5]. It is therefore expected that imposing the constraints (5.3) on the coordinates of the full coset $O(4,3)/(O(4)\times O(3))$ one should obtain a 7×7 representative of the $G2$ coset different from our previous one. This new $G2$ matrix could also be obtained as in [5, 16] by direct exponentiation of the Borel subalgebra using the new representation (5.4), (B.3)-(B.4) of the $g2$ algebra. Alternatively, one can as we have shown use the full new matrix (4.9) together with the corresponding transformations (5.4) to generate from a given seed a new solution of five-dimensional minimal supergravity.

At the same time, one can also transform solutions of MSG5 to non-trivial solutions of MSG6 by performing $O(4,3)$ transformations which do not belong to the $G_{2(2)}$ subgroup, which may be chosen to have required asymptotic properties, as discussed at the end of section 4. The same arguments equally apply to the $SU(2,1)/S(U(2)\times U(1))$ subspace of the $G2$ coset.

Acknowledgments

The authors are grateful to Adel Bouchareb, Chiang-Mei Chen, Cédric Leygnac and Nicolai Scherbluck for fruitful collaboration. The work of subsection 5B was carried out in collaboration with Adel Bouchareb. D.G. is grateful to the LAPTH Annecy for hospitality in December 2012 while the paper was written. His work was supported in part by the RFBR grant 11-02-01371-a.

A Isometry algebra of the metric (4.6)

Fifteen obvious Killing vectors are:

$$M_a{}^b = 2\lambda_{ac}\frac{\partial}{\partial\lambda_{cb}} + \omega_a\frac{\partial}{\partial\omega_b} + \delta_a^b\omega_c\frac{\partial}{\partial\omega_c} - \chi^b\frac{\partial}{\partial\chi^a} + \delta_a^b\chi^c\frac{\partial}{\partial\chi^c}\,, \tag{1.1}$$

generating linear transformations in the three-Killing vector space,

$$N^a = \frac{\partial}{\partial\omega_a}\,, \tag{1.2}$$

generating translations of the "magnetic" coordinates ω_a, and

$$R_a = \frac{\partial}{\partial\chi^a} + \varepsilon_{abc}\chi^b\frac{\partial}{\partial\omega_c}\,, \tag{1.3}$$

generating gauge transformations of the χ^a.

Their commutation relations are

$$\left[M_a{}^b, M_c{}^d\right] = \delta_c^b M_a{}^d - \delta_a^d M_c{}^b\,, \tag{1.4}$$
$$\left[M_a{}^b, N^c\right] = -\delta_a^c N^b - \delta_a^b N^c\,, \tag{1.5}$$
$$\left[M_a{}^b, R_c\right] = \delta_c^b R_a - \delta_a^b R_c\,, \tag{1.6}$$
$$\left[N^a, N^b\right] = 0\,, \tag{1.7}$$
$$[N^a, R_b] = 0\,, \tag{1.8}$$
$$[R_a, R_b] = -2\varepsilon_{abc}N^c\,. \tag{1.9}$$

Three more vectors L_a are needed to complete the algebra $sl(4,R)$ of the vacuum sector:

$$\left[M_a{}^b, L_c\right] = \delta_c^b L_a + \delta_a^b L_c\,, \tag{1.10}$$
$$[N^a, L_b] = M_b{}^a\,, \tag{1.11}$$
$$[L_a, L_b] = 0\,. \tag{1.12}$$

Adding to the known form of the $sl(4,R)$ for 6D Einstein the information from (1.11),

$$L_a = \omega_a\omega_b\frac{\partial}{\partial\omega_b} + 2\omega_b\lambda_{ac}\frac{\partial}{\partial\lambda_{bc}} + \chi^b(\omega_a\frac{\partial}{\partial\chi^b} - \omega_b\frac{\partial}{\partial\chi^a}) + \tau\lambda_{ab}\frac{\partial}{\partial\omega_b} + \cdots \tag{1.13}$$

(the omitted terms are of order 0 in ω_a). Assuming that the full Lie algebra is $O(4,3)$, it must close with the three remaining generators P^a defined by

$$[R_a, L_b] = \varepsilon_{abc} P^c \,, \tag{1.14}$$

leading to

$$P^a = \omega_b (\chi^b \frac{\partial}{\partial \omega_a} - \chi^a \frac{\partial}{\partial \omega_b} - \varepsilon^{abc} \frac{\partial}{\partial \chi_c}) + \cdots \,, \tag{1.15}$$

and obeying the commutation relations

$$\begin{aligned}
\left[M_a{}^b, P^c \right] &= -\delta_a^c P^b + \delta_a^b P^c \,, && (1.16)\\
\left[N^a, P^b \right] &= \varepsilon^{abc} R_c \,, && (1.17)\\
\left[R_a, P^b \right] &= 2 M_a{}^b - \delta_a^b \mathrm{Tr}(M) \,, && (1.18)\\
\left[L_a, P^b \right] &= 0 \,, && (1.19)\\
\left[P^a, P^b \right] &= -2 \varepsilon^{abc} L_c \,. && (1.20)
\end{aligned}$$

The degrees of the various fields can be found from their commutators with $\mathrm{Tr}(M)$;

$$[\lambda] = 2 \,, \quad [\omega] = 4 \,, \quad [\chi] = 2 \,. \tag{1.21}$$

This leads to the degrees of the various Killing vectors

$$\left[M_a{}^b \right] = 0 \,, \quad [R_a] = -2 \,, \quad \left[P^b \right] = 2 \,, \quad \left[N^b \right] = -4 \,, \quad [L_a] = 4 \,. \tag{1.22}$$

The six unknown Killing vectors L_a and P^a can be determined, up to a sign, by solving the commutation relations (1.18) and (1.20). The relatively simple result is

$$\begin{aligned}
L_a &= \omega_a \omega_b \frac{\partial}{\partial \omega_b} + 2 \omega_b \lambda_{ac} \frac{\partial}{\partial \lambda_{bc}} + \chi^b \left(\omega_a \frac{\partial}{\partial \chi^b} - \omega_b \frac{\partial}{\partial \chi^a} \right) + \tau \lambda_{ab} \frac{\partial}{\partial \omega_b} \\
&\quad - 2 \varepsilon_{abc} \chi^b \chi^d \lambda_{de} \frac{\partial}{\partial \lambda_{ec}} - \alpha \tau \varepsilon_{abc} \lambda^{bd} \chi^c \left(\frac{\partial}{\partial \chi^d} - \varepsilon_{def} \chi^e \frac{\partial}{\partial \omega_f} \right) \,, && (1.23)\\
P^a &= \omega_b \left(\chi^b \frac{\partial}{\partial \omega_a} - \chi^a \frac{\partial}{\partial \omega_b} - \varepsilon^{abc} \frac{\partial}{\partial \chi_c} \right) + 2 \chi^b \left(2 \lambda_{bc} \frac{\partial}{\partial \lambda_{ca}} - \delta_b^a \lambda_{dc} \frac{\partial}{\partial \lambda_{cd}} \right) \\
&\quad - \chi^a \chi^b \frac{\partial}{\partial \chi_b} - \alpha \tau \lambda^{ab} \left(\frac{\partial}{\partial \chi^b} - \varepsilon_{bcd} \chi^c \frac{\partial}{\partial \omega_d} \right) \,, && (1.24)
\end{aligned}$$

with $\alpha^2 = 1$.

The value of $\alpha = \pm 1$ is presumably related to the signature of λ (here $-++$). It can be determined by enforcing that e.g. $\gamma_a P^a$ (γ_a constant vector) is a Killing vector of the target space metric. The action of $(P\gamma)$ leads to the first order variations (written in matrix notation)

$$\begin{aligned}
\delta \lambda &= 2 \left[\gamma \cdot \chi \lambda + \lambda \chi \cdot \gamma - (\chi \gamma) \lambda \right] \,, \\
\delta \omega &= \gamma (\chi \omega) - (\chi \gamma) \omega + \alpha \tau \lambda^{-1} \gamma \wedge \chi \,, \\
\delta \chi &= -\gamma \wedge \omega - (\chi \gamma) \chi - \alpha \tau \lambda^{-1} \gamma \,.
\end{aligned} \tag{1.25}$$

This leads to

$$\delta(dl^2) = 4(1-\alpha)\left[(d\chi d\lambda\lambda^{-1}\gamma) - \tau^{-1}(d\tau + (\chi\lambda d\chi))(d\chi\gamma) + \tau^{-1}(\chi\gamma)(d\chi\lambda d\chi) - \tau^{-1}(\gamma,\lambda d\chi, d\omega)\right], \tag{1.26}$$

which vanishes provided

$$\alpha = +1\,. \tag{1.27}$$

B Matrix representative

The first step is to construct real matrix representatives of $O(4,3)$, beginning with the subalgebra $O(3,3) \sim sl(4,R)$. Rather than using the Maison parametrisation of $sl(4,R)$ in terms of 4×4 matrices (which presumably would lead to a representation of $O(4,3)$ in terms of 8×8 matrices), we use the representation of $O(3,3)$ in terms of 6×6 matrices, decomposed in 3×3 blocks according to

$$M_a{}^b = \begin{pmatrix} m_a{}^b & 0 \\ 0 & -\tilde{m}_a^b \end{pmatrix}, \quad N^a = \begin{pmatrix} 0 & n^a \\ 0 & 0 \end{pmatrix}, \quad L_a = \begin{pmatrix} 0 & 0 \\ -n^{aT} & 0 \end{pmatrix}, \tag{B.1}$$

where $\tilde{}$ denotes the anti-transposition, i.e. transposition relative to the anti- (or minor) diagonal, and

$$(m_a{}^b)^\alpha{}_\beta = \delta_a^\alpha \delta_\beta^b - \delta_a^b \delta_\beta^\alpha\,, \tag{B.2}$$

$$n^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad n^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$(\alpha,\beta = 1,2,3)$. These matrices satisfy the commutation relations (1.4), (1.5), (1.10), (1.7), (1.11) and (1.12).

The 7×7 matrix generators of $O(4,3)$ contain the preceding, promoted to 7×7 matrices by the addition of a central 3-row and a central 3-column, in block form

$$M_a{}^b = \begin{pmatrix} m_a{}^b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tilde{m}_a^b \end{pmatrix}, \quad N^a = \begin{pmatrix} 0 & 0 & n^a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -n^{aT} & 0 & 0 \end{pmatrix}, \tag{B.3}$$

together with

$$R_a = \sqrt{2}\begin{pmatrix} 0 & r_a & 0 \\ 0 & 0 & -\tilde{r}_a \\ 0 & 0 & 0 \end{pmatrix}, \quad P^a = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ r_a^T & 0 & 0 \\ 0 & -\tilde{r}_a^T & 0 \end{pmatrix}, \tag{B.4}$$

where r_a is the column matrix of elements

$$r_a^\alpha = \delta_a^\alpha\,. \tag{B.5}$$

Using

$$r_a \tilde{r}_b - r_b \tilde{r}_a = \varepsilon_{abc} n^c\,, \tag{B.6}$$

these can be checked to satisfy the remaining commutation relations of $O(4,3)$.

The 7×7 coset matrix representative is

$$\mathcal{M} = \mathcal{V}^T \mathcal{M}_0 \mathcal{V}\,, \tag{B.7}$$

with

$$\mathcal{M}_0 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \tilde{\mu}^{-1} \end{pmatrix}, \quad \mu = \tau^{-1}\lambda\,, \tag{B.8}$$

and

$$\mathcal{V} = e^{\chi^a R_a} e^{\omega_a N^a} = \begin{pmatrix} 1 & \sqrt{2}\chi & \gamma \\ 0 & 1 & -\sqrt{2}\tilde{\chi} \\ 0 & 0 & 1 \end{pmatrix}, \tag{B.9}$$

where

$$\chi = \begin{pmatrix} \chi^1 \\ \chi^2 \\ \chi^3 \end{pmatrix}, \quad \gamma = \hat{\omega} - \chi\tilde{\chi}\,, \quad \hat{\omega} = \begin{pmatrix} -\omega_2 & \omega_3 & 0 \\ \omega_1 & 0 & -\omega_3 \\ 0 & -\omega_1 & \omega_2 \end{pmatrix}. \tag{B.10}$$

The resulting coset representative

$$\mathcal{M} = \begin{pmatrix} \mu & \sqrt{2}\mu\chi & \mu\gamma \\ \sqrt{2}\chi^T\mu & -1+2\chi^T\mu\chi & \sqrt{2}(\chi^T\mu\gamma+\tilde{\chi}) \\ \gamma^T\mu & \sqrt{2}(\gamma^T\mu\chi+\tilde{\chi}^T) & \gamma^T\mu\gamma - 2\tilde{\chi}^T\tilde{\chi}+\tilde{\mu}^{-1} \end{pmatrix} \tag{B.11}$$

is related to its inverse by

$$\mathcal{M}^{-1} = \tilde{\mathcal{M}} \tag{B.12}$$

(use $\tilde{\mathcal{V}}(\omega,\chi) = \mathcal{V}(-\omega,-\chi)$). Taking into account the identity

$$\mathrm{Tr}[\tilde{\lambda} V^T \lambda V] = -2\tau(V^T\lambda^{-1}V)\,,$$

which follows from (B.14), one checks that the target space metric (4.6) can be expressed as

$$dl^2 = \frac{1}{4}\mathrm{Tr}(\mathcal{M}^{-1}d\mathcal{M}\mathcal{M}^{-1}d\mathcal{M})\,. \tag{B.13}$$

The Kaluza-Klein vectors a_i^a can be recovered directly by solving the duality equation (4.5), where the field V is contained in the block

$$\mathcal{J}_{31} = \tau^{-2}\tilde{\lambda}(d\hat{\omega} + d\chi\tilde{\chi} - \chi d\tilde{\chi})^T\lambda = -\tau^{-1}(\widehat{\lambda^{-1}V})^T \tag{B.14}$$

of the current

$$\mathcal{J} = \mathcal{M}^{-1}d\mathcal{M} \tag{B.15}$$

(with the hat ˆ vector-to-matrix transformation defined as in the last equation (B.10)).

References

[1] M. Gunaydin and F. Gürsey: J. Math. Phys. **14**, 1651 (1973).

[2] M. Gunaydin, A. Neitzke, O. Pavlyk and B. Pioline, Commun. Math. Phys. **283**, 169 (2008) [arXiv:0707.1669 [hep-th]].

[3] E. Cremmer, B. Julia, H. Lu and C. N. Pope, "Higher dimensional origin of D = 3 coset symmetries", hep-th/9909099.

[4] S. Mizoguchi and N. Ohta, Phys. Lett. B **441** (1998) 123 [hep-th/9807111].

[5] A. Bouchareb, G. Clément, C. -M. Chen, D. V. Gal'tsov, N. G. Scherbluk, T. Wolf, Phys. Rev. **D76**, 104032 (2007). [arXiv:0708.2361 [hep-th]].

[6] F. J. Ernst, Phys. Rev. **167**, 1175 (1968); F. J. Ernst, Phys. Rev. **168**, 1415 (1968).

[7] W. Kinnersley, J. Math. Phys. 14: 651-653 (1973); J. Math. Phys. **18**, 1529 (1977). [hep-th/9909099].

[8] P. Breitenlohner, D. Maison, G. W. Gibbons, Commun. Math. Phys. **120**, 295 (1988).

[9] D. Maison, Lect. Notes Phys. **540**, 273-323 (2000).

[10] G. Clément, "Sigma-model approaches to exact solutions in higher-dimensional gravity and supergravity", arXiv:0811.0691 [hep-th].

[11] D. V. Gal'tsov, Prog. Theor. Phys. Suppl. **172**, 121 (2008) [arXiv:0901.0098 [gr-qc]].

[12] P. O. Mazur, Acta Phys. Polon. B **14**, 219 (1983).

[13] S. Tomizawa, Y. Yasuii and Y. Morisawa, Class. Quantum Grav. **26**, 145006 (2009) [arXiv:0809.2001 [hep-th]].

[14] G. Compère, S. de Buyl, E. Jamsin and A. Virmani, Class. Quantum Grav. **26**, 125016 (2009) [arXiv:0903.1645 [hep-th]].

[15] G. Compère, S. de Buyl, S. Stotyn and A. Virmani, JHEP 1011:133 (2010) [arXiv:1006.5464 [hep-th]].

[16] G. Clément: J. Math. Phys. **49**, 042503 (2008), Erratum, J. Math. Phys. **49**, 079901 (2008) [arXiv:0710.1192[gr-qc]]

[17] S. F. Hassan and A. Sen, Nucl. Phys. B **375**, 103 (1992) [hep-th/9109038].

[18] J. Maharana and J. H. Schwarz, Nucl. Phys. B **390**, 3 (1993) [hep-th/9207016].

[19] A. Sen, Int. J. Mod. Phys. A **9**, 3707 (1994) [hep-th/9402002].

[20] P. Pasti, D. P. Sorokin and M. Tonin, Phys. Rev. D **55**, 6292 (1997) [hep-th/9611100]; G. Dall'Agata, K. Lechner and M. Tonin, Nucl. Phys. B **512** (1998) 179 [hep-th/9710127].

[21] A. Sen, Nucl. Phys. B **434**, 179 (1995) [hep-th/9408083].

[22] D. Maison, Gen. Rel. Grav. **10**, 717-723 (1979).

[23] F.J. Ernst and W.J. Wild, J. Math. Phys. **17**, 182 (1976).

[24] G. Clément and D. V. Gal'tsov, Phys. Rev. D , (2013)

[25] R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications*, John Wiley & Sons, 1974.